MATH2050C Assignment 5

Deadline: Feb 14, 2018.

Hand in: 3.2 no. 14, 19d; 3.3 no. 5, 10; Supplementary Exercise (2).

Section 3.2 no. 14, 17, 18, 19, 21.

Section 3.3 no. 1, 3, 5, 7, 10, 11, 12.

Supplementary Exercises

1. Show that
$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n$$
exists for every $a > 0$.

2. Consider $\{x_n\}$ where
$$x_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \ , \quad n \geq 1 \ .$$
Show that
$$\lim_{n\to\infty} x_n = e \ .$$

5.1 Two Most Commonly Used Results in Limits of Sequences

Most limits of sequences can be obtained based on the following two theorems.

Theorem 1 (Limit Theorem). Let $\{a_n\}, \{b_n\}$ be two convergent sequences with $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then

1. The sequence $\{\alpha a_n + \beta b_n\}$ is convergent and
$$\lim_{n\to\infty} (\alpha a_n + \beta b_n) = \alpha a + \beta b.$$
2. The sequence $\{a_n b_n\}$ is convergent and
$$\lim_{n\to\infty} a_n b_n = ab\ .$$
3. In case $b_n, b \neq 0$, the sequence $\{a_n/b_n\}$ is convergent and
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}\ .$$

In (3), as $b \neq 0$, there is some n_0 such that $b_n \neq 0$ for all $n \geq n_0$. (It suffices to fix n_0 such that $|b_n - b| < |b|/2$ for all $n \geq n_0$.) The assumption $b_n \neq 0$ follows from $b \neq 0$ if we consider the quotient sequence as a sequence beginning from the n_0-th term, or its n_0-th tail. Obviously it does no harm as the notion of the limit is concerned with "limiting behavior".

This theorem shows that limits of sequences behave very nicely under algebraic operations.

Theorem 2 (Squeeze Theorem). Let $a_n \leq c_n \leq b_n$ for the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ for all $n \geq n_0$. In case $a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ exists, then $\{c_n\}$ is convergent and $\lim_{n\to\infty} c_n = a$ too.

It is also called the **Sandwich Rule**. In applications, it is very often that $\{a_n\}$ is the zero sequence, $\{c_n\} = \{|x_n - x|\}$ and $\{b_n\}, b_n \to 0$, so that we can conclude $\lim_{n\to\infty} x_n = x$. The Sandwich Rule enables us to simplify the expression of the sequence under consideration.

5.2 Some Examples of Limits

We summarize some basic and non-trivial results on how fast some sequences go to infinity in Theorem 3. In the following we use the notation
$$\{a_n\} << \{b_n\}, \quad \text{or } a_n << b_n\ ,$$
to mean
$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0\ ,$$
for the sequences $\{a_n\}$ and $\{b_n\}$.

Theorem 3. For each $k \geq 1$ and $a > 1$,
$$n^k << a^n << n! << n^n\ .$$

Proof. First, $\lim_{n\to\infty} \dfrac{n^k}{a^n} = 0$. We use Ration Test

$$\frac{(n+1)^k/a^{n+1}}{n^k/a^n} = \left(1+\frac{1}{n}\right)^k \frac{1}{a} \to \frac{1}{a} < 1 \ ,$$

as $n \to \infty$. Fix a number $r \in (1/a, 1)$, we can find a large n_0 such that this quotient is less than r for all $n \geq n_0$. The desired result follows from the Ratio Test.

Next, $\lim_{n\to\infty} \dfrac{a^n}{n!} = 0$. Again this follows from the Ratio Test.

Third, for $n \geq 2$,

$$\frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\cdots\frac{n}{n} \leq \frac{1}{n} \ ,$$

thus

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

by Squeeze Theorem.

We also note the following limits.

Theorem 4. For $a > 0$,

$$\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} n^{1/n} = 1 \ .$$

Proof. It is clear that $n^{1/n} > 1$ for $n \geq 2$. Write $n^{1/n} = 1 + k_n$ and, by Binomial Theorem,

$$n = (1+k_n)^n = \sum_{j=0}^{n} \tbinom{n}{j} k_n^j \geq \tbinom{n}{2} k_n^2 = \frac{n(n-1)}{2} k_n^2 \ .$$

Consequently,

$$\frac{2}{n-1} \geq k_n^2 \ , \quad \text{or } k_n \leq \sqrt{\frac{2}{n-1}},$$

which tends to 0 as $n \to \infty$. Hence

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} (1+k_n) = 1 \ .$$

When $a > 1$, $a^{1/n} > 1$ for all n. For $n \geq a$, $1 \leq a^{1/n} \leq n^{1/n}$ and it follows from Squeeze Theorem that

$$\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} n^{1/n} = 1 \ .$$

When $a < 1$, $b = 1/a > 1$ and by Limit Theorem

$$\lim_{n\to\infty} a^{1/n} = \frac{1}{\lim_{n\to\infty} b^{1/n}} = 1 \ .$$

When $a = 1$, the result is trivial. Everything done.